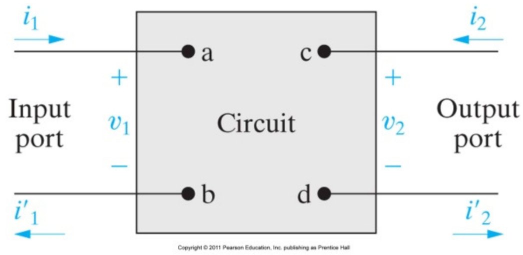
Chapter 18: Two-Port Circuits



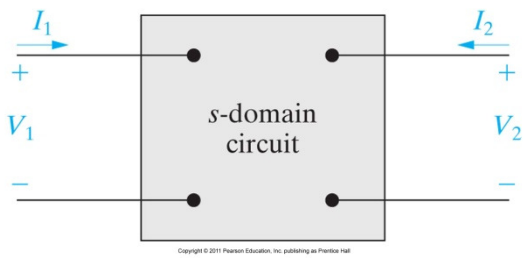
The two-port model is used to describe the performance of a circuit in terms of the voltage and current at its input and output ports.

Model Restrictions

- No independent sources are inside the circuit between the ports (dependent can be)
- No energy is stored inside the circuit between the ports
- The current into the port is equal to the current out of the port $i_1 = i_1'$ & $i_2 = i_2'$
- No external connections exist between the input and output ports

Thus for a two-port circuit only the terminal voltages and currents are of concern

18.1 The Terminal Equations



Mostly carried out in the s-domain

Current and voltage symmetries

- Current is directed into the top ports
- Voltages rise from lower to upper

Two of the four terminal variables (I_1, I_2, V_1, V_2) are independent therefore only two simultaneous equations are needed to describe the circuit. But there are six different was to describe these equations.

$$V_1 = z_{11}I_1 + z_{12}I_2 \qquad I_1 = y_{11}V_1 + y_{12}V_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2 \qquad I_2 = y_{21}V_1 + y_{22}V_2$$

$$V_1 = a_{11}V_2 - a_{12}I_2 \qquad V_2 = b_{11}V_1 - b_{12}I_1$$
$$I_1 = a_{21}V_2 - a_{22}I_2 \qquad I_2 = b_{21}V_1 - b_{22}I_1$$

$$V_1 = h_{11}I_1 + h_{12}V_2 \qquad I_1 = g_{11}V_1 + g_{12}I_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2 \qquad V_2 = g_{21}V_1 + g_{22}I_2$$

Where z, g, a, b, h and g are the **parameters** of the network.

18.2 The Two-Port Parameters

The values of a network can be determined by either computation or measurement.

To compute the z parameters:

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \Omega \qquad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \Omega \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \Omega \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \Omega$$

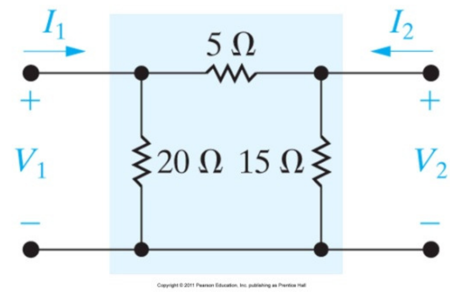
Measuring the z parameters

- z_{11} is the impedance seen looking into port 1 when port 2 is open
- z_{12} is a transfer impedance. It is the ratio of port 1 voltage to port 2 current when port 1 is open.
- z_{21} is a transfer impedance. It is the ratio of port 2 voltage to port 1current when port 2 is open.
- z_{22} is the impedance seen looking into port 2 when port 1 is open

Note: each z parameter is a ratio of voltage over current and in ohms, thus its name

Example 18.1

Find the z parameters of the following circuit.



Each of the remaining parameters can be found in a similar way.

$$y_{11} = \left.\frac{I_1}{V}\right|_{V_2=0} \text{ S} \qquad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \text{ S} \qquad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \text{ S} \qquad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \text{ S}$$

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0} \Omega \qquad a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0} \text{ S} \qquad a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

$$b_{11} = \left.\frac{V_2}{V_1}\right|_{I_1=0} \qquad b_{12} = -\left.\frac{V_2}{I_1}\right|_{V_1=0} \Omega \qquad b_{21} = \left.\frac{I_2}{V_1}\right|_{I_1=0} \text{ S} \qquad b_{22} = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \Omega \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \text{ S}$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \text{ S} \qquad g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} \qquad g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} \qquad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} \Omega$$

The reciprocal set of parameters for impedance (z) and admittance (y) and called the **immittance** parameters.

The reciprocal set of parameters a and b are called the **transmission** parameters because they describe the voltage and current at one end in terms of the voltage and current at the other end.

Parameter h and g relate cross-variables input voltage and output current and output voltage to input current and are called the **hybrid** parameters.

Example 18.2 and Assessment Problems 18.1 – 18.3

Relationship between the Two-Port Parameters

TABLE 18.1 Parameter Conversion Table

$$z_{11} = \frac{y_{22}}{\Delta y} = \frac{a_{11}}{a_{21}} = \frac{b_{22}}{b_{21}} = \frac{\Delta h}{h_{22}} = \frac{1}{g_{11}}$$

$$z_{12} = -\frac{y_{12}}{\Delta y} = \frac{\Delta a}{a_{21}} = \frac{1}{b_{21}} = \frac{h_{12}}{h_{22}} = -\frac{g_{12}}{g_{11}}$$

$$z_{21} = \frac{-y_{21}}{\Delta y} = \frac{1}{a_{21}} = \frac{\Delta b}{b_{21}} = -\frac{h_{21}}{h_{22}} = \frac{g_{21}}{g_{11}}$$

$$z_{22} = \frac{y_{11}}{\Delta y} = \frac{a_{22}}{a_{21}} = \frac{b_{11}}{b_{21}} = \frac{1}{h_{22}} = \frac{\Delta g}{g_{11}}$$

$$y_{11} = \frac{z_{22}}{\Delta z} = \frac{a_{22}}{a_{12}} = \frac{b_{11}}{b_{12}} = \frac{1}{h_{11}} = \frac{\Delta g}{g_{22}}$$

$$y_{12} = -\frac{z_{12}}{\Delta z} = -\frac{\Delta a}{a_{12}} = -\frac{1}{b_{12}} = -\frac{h_{12}}{h_{11}} = \frac{g_{12}}{g_{22}}$$

$$y_{21} = -\frac{z_{21}}{\Delta z} = -\frac{1}{a_{12}} = -\frac{\Delta b}{b_{12}} = \frac{h_{21}}{h_{11}} = -\frac{g_{21}}{g_{22}}$$

$$b_{21} = \frac{1}{z_{12}} = -\frac{\Delta y}{y_{12}} = \frac{a_{21}}{\Delta a} = \frac{h_{22}}{h_{12}} = -\frac{g_{11}}{g_{12}}$$

$$b_{22} = \frac{z_{11}}{z_{12}} = \frac{y_{22}}{y_{12}} = \frac{a_{11}}{\Delta a} = \frac{\Delta h}{h_{12}} = -\frac{1}{g_{12}}$$

$$h_{11} = \frac{\Delta z}{z_{22}} = \frac{1}{y_{11}} = \frac{a_{12}}{a_{22}} = \frac{b_{12}}{b_{11}} = \frac{g_{22}}{\Delta g}$$

$$h_{12} = \frac{z_{12}}{z_{22}} = -\frac{y_{12}}{y_{11}} = \frac{\Delta a}{a_{22}} = \frac{1}{b_{11}} = -\frac{g_{12}}{\Delta g}$$

$$h_{21} = -\frac{z_{21}}{z_{22}} = \frac{y_{21}}{y_{11}} = -\frac{1}{a_{22}} = -\frac{\Delta b}{b_{11}} = -\frac{g_{21}}{\Delta g}$$

$$h_{22} = \frac{1}{z_{22}} = \frac{\Delta y}{y_{11}} = \frac{a_{21}}{a_{22}} = \frac{b_{21}}{b_{11}} = \frac{g_{11}}{\Delta g}$$

$$g_{11} = \frac{1}{z_{11}} = \frac{\Delta y}{y_{22}} = \frac{a_{21}}{a_{11}} = \frac{b_{21}}{b_{22}} = \frac{h_{22}}{\Delta h}$$

TABLE 18.1 Parameter Conversion Table

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{a_{11}}{a_{12}} = \frac{b_{22}}{b_{12}} = \frac{\Delta h}{h_{11}} = \frac{1}{g_{22}}$$

$$a_{11} = \frac{z_{11}}{z_{21}} = -\frac{y_{22}}{y_{21}} = \frac{b_{22}}{\Delta b} = -\frac{\Delta h}{h_{21}} = \frac{1}{g_{21}}$$

$$a_{12} = \frac{\Delta z}{z_{21}} = -\frac{1}{y_{21}} = \frac{b_{12}}{\Delta b} = -\frac{h_{11}}{h_{21}} = \frac{g_{22}}{g_{21}}$$

$$a_{21} = \frac{1}{z_{21}} = -\frac{\Delta y}{y_{21}} = \frac{b_{21}}{\Delta b} = -\frac{h_{22}}{h_{21}} = \frac{g_{11}}{g_{21}}$$

$$a_{22} = \frac{z_{22}}{z_{21}} = -\frac{y_{11}}{y_{21}} = \frac{b_{11}}{\Delta b} = -\frac{1}{h_{21}} = \frac{\Delta g}{g_{21}}$$

$$b_{11} = \frac{z_{22}}{z_{12}} = -\frac{y_{11}}{y_{12}} = \frac{a_{22}}{\Delta a} = \frac{1}{h_{12}} = -\frac{\Delta g}{g_{12}}$$

$$b_{12} = \frac{\Delta z}{z_{12}} = -\frac{1}{y_{12}} = \frac{a_{12}}{\Delta a} = \frac{h_{11}}{h_{12}} = -\frac{g_{22}}{g_{12}}$$

$$g_{12} = -\frac{z_{12}}{z_{11}} = \frac{y_{12}}{y_{22}} = -\frac{\Delta a}{a_{11}} = -\frac{1}{b_{22}} = -\frac{h_{12}}{\Delta h}$$

$$g_{21} = \frac{z_{21}}{z_{11}} = -\frac{y_{21}}{y_{22}} = \frac{1}{a_{11}} = \frac{\Delta b}{b_{22}} = -\frac{h_{21}}{\Delta h}$$

$$g_{22} = \frac{\Delta z}{z_{11}} = \frac{1}{y_{22}} = \frac{a_{12}}{a_{11}} = \frac{b_{12}}{b_{22}} = \frac{h_{11}}{\Delta h}$$

$$\Delta z = z_{11}z_{22} - z_{12}z_{21}$$

$$\Delta y = y_{11}y_{22} - y_{12}y_{21}$$

$$\Delta a = a_{11}a_{22} - a_{12}a_{21}$$

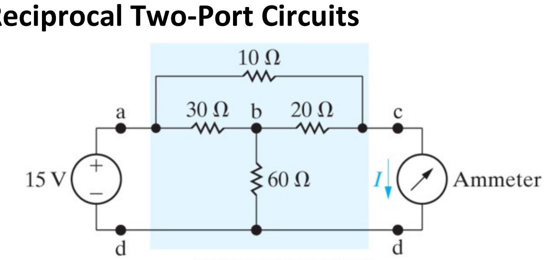
$$\Delta b = b_{11}b_{22} - b_{12}b_{21}$$

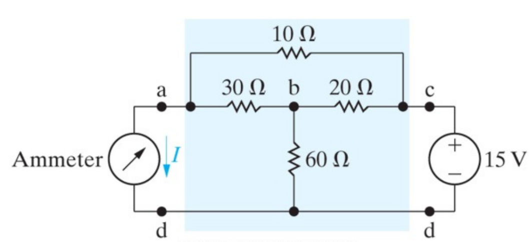
$$\Delta h = h_{11}h_{22} - h_{12}h_{21}$$

$$\Delta g = g_{11}g_{22} - g_{12}g_{21}$$

Example 18.3 and Assessment Problems 18.4

Reciprocal Two-Port Circuits





A two-port circuit is reciprocal if the interchange of an ideal voltage source at one port with an ideal ammeter at the other port produces the same ammeter reading.

A two-port circuit is also a reciprocal if the interchange of an ideal current source at one port with an ideal voltmeter at the other port produces the same voltmeter reading.

Relationships for a reciprocal two-port circuit

$$z_{12} = z_{21} \qquad a_{11}a_{22} - a_{12}a_{21} = \Delta a = 1 \qquad h_{12} = -h_{21}$$
$$y_{12} = y_{21} \qquad b_{11}b_{22} - b_{12}b_{21} = \Delta \text{b} = 1 \qquad g_{12} = -g_{21}$$

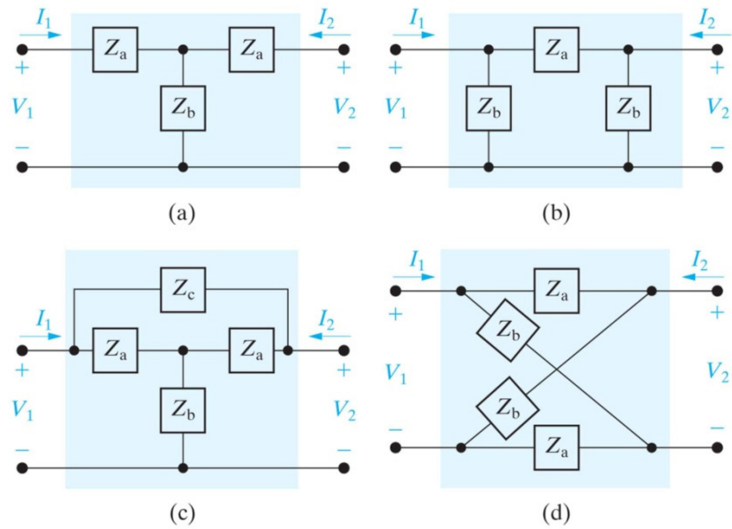
A two-port circuit is symmetric if its ports can be interchanged without disturbing the values of the terminal currents and voltages.

Relationships for a symmetric two-port circuit

$$z_{11} = z_{22}$$
$$y_{11} = y_{22}$$

$$a_{11} = a_{22}$$
$$b_{11} = b_{22}$$

$$h_{11}h_{22} - h_{12}h_{21} = \Delta \text{h} = 1$$
$$g_{11}g_{22} - g_{12}g_{21} = \Delta \text{b} = 1$$

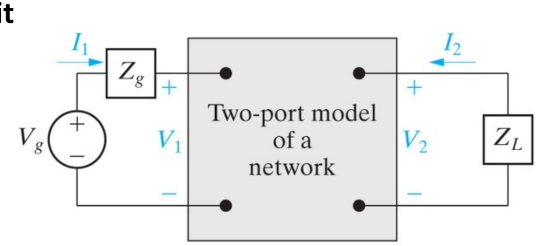


Assessment Problem 18.5

18.3 Analysis of the Terminated Two-Port Circuit

Typical two-port circuits are driven in port 1 and loaded on port 2.

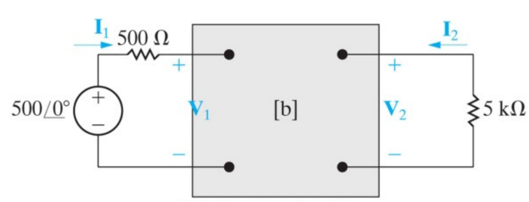
Where V_g & Z_g are the voltage and impedance of the source. Z_L is the load impedance.



6 characteristics of a terminated two-port circuit

- Input impedance $Z_{in} = V_1 / I_1$ or the admittance $Y_{in} = I_1 / V_1$
- The output current I_2
- The Thevenin voltage and impedance (V_{Th} & Z_{Th}) with respect to port 2
- The current gain I_2 / I_1
- The voltage gain V_2 / V_1
- The voltage gain V_2 / V_g

See the following page for the appropriate equations



Example 18.4
Assessment Problem 18.6

TABLE 18.2 Terminated Two-Port Equations

z Parameters	*y* Parameters
$Z_{in} = z_{11} - \dfrac{z_{12}z_{21}}{z_{22} + Z_L}$	$Y_{in} = y_{11} - \dfrac{y_{12}y_{21}Z_L}{1 + y_{22}Z_L}$
$I_2 = \dfrac{-z_{21}V_g}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}}$	$I_2 = \dfrac{y_{21}V_g}{1 + y_{22}Z_L + y_{11}Z_g + \Delta y Z_g Z_L}$
$V_{Th} = \dfrac{z_{21}}{z_{11} + Z_g}V_g$	$V_{Th} = \dfrac{-y_{21}V_g}{y_{22} + \Delta y Z_g}$
$Z_{Th} = z_{22} - \dfrac{z_{12}z_{21}}{z_{11} + Z_g}$	$Z_{Th} = \dfrac{1 + y_{11}Z_g}{y_{22} + \Delta y Z_g}$
$\dfrac{I_2}{I_1} = \dfrac{-z_{21}}{z_{22} + Z_L}$	$\dfrac{I_2}{I_1} = \dfrac{y_{21}}{y_{11} + \Delta y Z_L}$
$\dfrac{V_2}{V_1} = \dfrac{z_{21}Z_L}{z_{11}Z_L + \Delta z}$	$\dfrac{V_2}{V_1} = \dfrac{-y_{21}Z_L}{1 + y_{22}Z_L}$
$\dfrac{V_2}{V_g} = \dfrac{z_{21}Z_L}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}}$	$\dfrac{V_2}{V_g} = \dfrac{y_{21}Z_L}{y_{12}y_{21}Z_g Z_L - (1 + y_{11}Z_g)(1 + y_{22}Z_L)}$

TABLE 18.2 Terminated Two-Port Equations

a Parameters	*b* Parameters
$Z_{in} = \dfrac{a_{11}Z_L + a_{12}}{a_{21}Z_L + a_{22}}$	$Z_{in} = \dfrac{b_{22}Z_L + b_{12}}{b_{21}Z_L + b_{11}}$
$I_2 = \dfrac{-V_g}{a_{11}Z_L + a_{12} + a_{21}Z_g Z_L + a_{22}Z_g}$	$I_2 = \dfrac{-V_g \Delta b}{b_{11}Z_g + b_{21}Z_g Z_L + b_{22}Z_L + b_{12}}$
$V_{Th} = \dfrac{V_g}{a_{11} + a_{21}Z_g}$	$V_{Th} = \dfrac{V_g \Delta b}{b_{22} + b_{21}Z_g}$
$Z_{Th} = \dfrac{a_{12} + a_{22}Z_g}{a_{11} + a_{21}Z_g}$	$Z_{Th} = \dfrac{b_{11}Z_g + b_{12}}{b_{21}Z_g + b_{22}}$
$\dfrac{I_2}{I_1} = \dfrac{-1}{a_{21}Z_L + a_{22}}$	$\dfrac{I_2}{I_1} = \dfrac{-\Delta b}{b_{11} + b_{21}Z_L}$
$\dfrac{V_2}{V_1} = \dfrac{Z_L}{a_{11}Z_L + a_{12}}$	$\dfrac{V_2}{V_1} = \dfrac{\Delta b Z_L}{b_{12} + b_{22}Z_L}$
$\dfrac{V_2}{V_g} = \dfrac{Z_L}{(a_{11} + a_{21}Z_g)Z_L + a_{12} + a_{22}Z_g}$	$\dfrac{V_2}{V_g} = \dfrac{\Delta b Z_L}{b_{12} + b_{11}Z_g + b_{22}Z_L + b_{21}Z_g Z_L}$

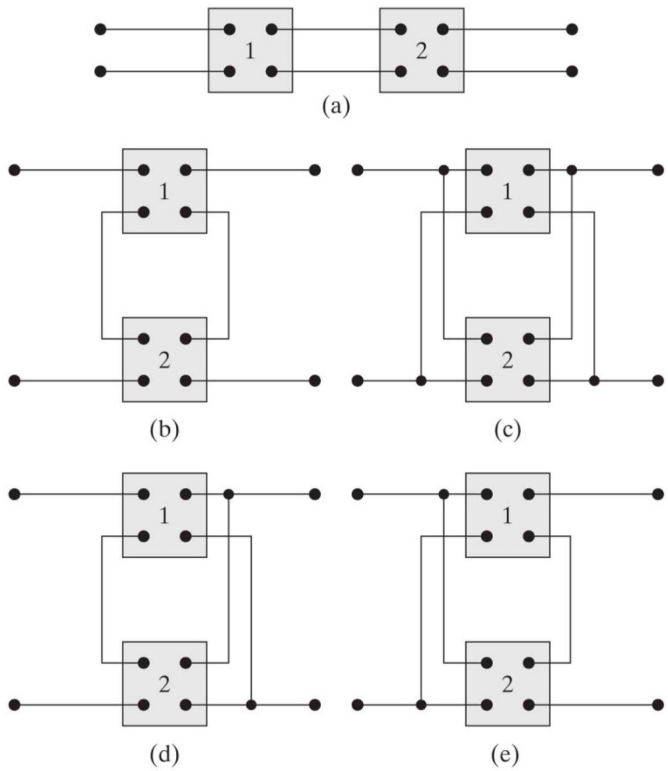
TABLE 18.2 Terminated Two-Port Equations

h Parameters	*g* Parameters
$Z_{in} = h_{11} - \dfrac{h_{12}h_{21}Z_L}{1 + h_{22}Z_L}$	$Y_{in} = g_{11} - \dfrac{g_{12}g_{21}}{g_{22} + Z_L}$
$I_2 = \dfrac{h_{21}V_g}{(1 + h_{22}Z_L)(h_{11} + Z_g) - h_{12}h_{21}Z_L}$	$I_2 = \dfrac{-g_{21}V_g}{(1 + g_{11}Z_g)(g_{22} + Z_L) - g_{12}g_{21}Z_g}$
$V_{Th} = \dfrac{-h_{21}V_g}{h_{22}Z_g + \Delta h}$	$V_{Th} = \dfrac{g_{21}V_g}{1 + g_{11}Z_g}$
$Z_{Th} = \dfrac{Z_g + h_{11}}{h_{22}Z_g + \Delta h}$	$Z_{Th} = g_{22} - \dfrac{g_{12}g_{21}Z_g}{1 + g_{11}Z_g}$
$\dfrac{I_2}{I_1} = \dfrac{h_{21}}{1 + h_{22}Z_L}$	$\dfrac{I_2}{I_1} = \dfrac{-g_{21}}{g_{11}Z_L + \Delta g}$
$\dfrac{V_2}{V_1} = \dfrac{-h_{21}Z_L}{\Delta h Z_L + h_{11}}$	$\dfrac{V_2}{V_1} = \dfrac{g_{21}Z_L}{g_{22} + Z_L}$
$\dfrac{V_2}{V_g} = \dfrac{-h_{21}Z_L}{(h_{11} + Z_g)(1 + h_{22}Z_L) - h_{12}h_{21}Z_L}$	$\dfrac{V_2}{V_g} = \dfrac{g_{21}Z_L}{(1 + g_{11}Z_g)(g_{22} + Z_L) - g_{12}g_{21}Z_g}$

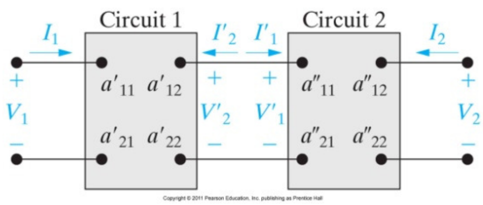
18.4 Interconnected Two-Port Circuits

5 ways to connect two-port circuits

a) Cascade (a-param)
b) Series (z-param)
c) Parallel (y-param)
d) Series-Parallel (h-param)
e) Parallel-Series (g-param)



We will focus on the cascade connection:



Where a' indicates first circuit. a'' the second.

V_2' and I_2' are the inputs to the second circuit V_1' and I_1'

The new a-parameters become

$$a_{11} = a'_{11}a''_{11} + a'_{12}a''_{21} \qquad a_{12} = a'_{11}a''_{12} + a'_{12}a''_{22}$$

$$a_{21} = a'_{21}a''_{11} + a'_{22}a''_{21} \qquad a_{22} = a'_{21}a''_{12} + a'_{22}a''_{22}$$

If more than two units are connected in cascade, the a-parameters of the equivalent two-port circuit are found by successively reducing the circuits one pair at a time.

Example 18.5

Assessment Problem 18.7